Recognition of abstract angles in familiar physical situations

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A sample of 480 students from grades 2, 4, 6 and 8 were interviewed concerning their recognition of abstract angles in nine familiar physical situations. The major factor influencing students' interpretation of an angle situation was not the number of arms of the abstract angle visible in the physical situation. Even by the end of grade 8, many students had difficulty recognising angles in situations where both arms of the angle were not visible. Implications are drawn for the teaching of angles.

The development of an abstract concept can be considered to be the result of a classification process which starts with the identification of similarities between different experiences (Mitchelmore & White, 1995; White & Mitchelmore, 1996). A definition of any given concept is a succinct description of the predominant similarities in all the different situations from which the concept has been abstracted. In the case of angle, the definition given by the New South Wales K - 6 syllabus is "the amount of turning between two lines about a common point" (New South Wales Department of Education, 1989, p.79). This definition derives from the argument that the development of the angle concept consists of establishing a particular relationship between the two arms of the angle; and a turn is a familiar, physical action which relates the two arms (Wilson & Adams, 1992). For example, the angle of slope in a hill needs to be seen as the amount of turning between the horizontal and a line which gives the slope of the hill. Given that a hill does not involve physical turning and the horizontal line is not visible, the question arises as to how well students can actually visualise abstract angles in familiar physical situations. How well do they see the turn, the two lines and the common point?

There is little published research on students' interpretations of angles. Krainer (1989) reports that 12 year olds tended to give examples of angles which involved some sort of turning motion. Davey & Pegg (1991) found that, among students from Grades 1 to 10, descriptions of an angle went through a sequence of four stages: (1) a corner which is pointy or sharp; (2) a place where two lines meet; (3) the distance or area between two lines; and (4) the difference between the slopes of the lines. Clements & Battista (1989) report that grade 3 students commonly described an angle as a sloping line, a place where two lines meet, the two lines themselves, or a turn (the last only by students who had studied Logo). Such studies suggest that students do conceptualise their surroundings in terms of angles, and that their concept of angle develops in sophistication as they get older. However, these studies do not show how the developing angle concept is linked to children's physical experiences.

Recent studies (Mitchelmore, 1996; White & Mitchelmore, 1995) suggest, for example, that young children may not interpret many physical angle situations in terms of turning. In these studies, children in each of grades 2 and 4 were presented with models of a number of physical situations (a doll turning about a fixed axis, a variable hill, a pair of scissors, a map of some bending roads, a ball game and floor tiles). It was found that children often did not see the desired angle-related features in the situations, but focused on features arising from other specific details within the situation. In particular, angles occurring in the turning doll and rebounds in the ball game were generally seen as quite different to angles in the corners of tiles, slopes of a hill, bends in a road and opening/closing objects like scissors. Although the grade 4 children had all studied angles as an amount of turning in school, very few stated this definition of angle or gave turning as a example of an angle. Furthermore, about a quarter of the students explicitly stated that the turning doll situation did not involve angles.

Mitchelmore and White (1996b) specifically investigated how children from each of grades 2, 4 and 6 conceptualise and classify various turning situations. It was found that between grades 2 and 6 there was a general increase in children's tendency to recognise angle-related similarities between turning situations. However, relatively few children referred to the common movement in these situations as turning. In fact, children more often reported angle-related similarities based on the static appearance of the angles than ones based the dynamic way in which the angles were formed.

We have recently completed a study of how students in grades 2, 4, 6 and 8 interpret familiar physical situations in terms of angles. The grade 2 results were presented in Mitchelmore and White (1996a). The present paper reports on grades 4, 6 and 8. Some grade 2 responses have since been rescored for consistency with the analysis of the data from grades 4-8.

Method

Sample

The sample was gender-balanced and consisted of 144 children in each of grades 2, 4, 6 chosen from six schools in Sydney. A further 48 grade 8 students from two high schools were also interviewed.

Materials

Nine physical angle situations were used: wheel, door, scissors, hand fan, signpost, hill, road junction, tile and wall. The first four of these were movable while the last five were fixed. Each movable situation was represented by a single adjustable model. Each of the fixed situations was represented by a set of three models representing a "neutral" configuration (angle 0° or 90°), an angle of 45°, and a "middle" angle (22.5° or 67.5°). Adjustable models of the fixed situations were deliberately not used, in order to avoid suggesting a turning interpretation. Examples of the models used are shown in Figure 1.

A drinking straw which could be bent at various angles was used as an abstract angle model. A second straw fixed at 45° was also used.

Procedure

A trained research assistant administered interviews which investigated how well children modelled angles in the standard manner. The term "standard manner" refers to generally accepted angular modelling in which the vertex and the two arms of the abstract model match a point and two lines on the physical model. For the moveable situations, the vertex is the point of rotation and the opening of the arms represents the amount of rotation. For the fixed situations, the two arms represent two specific lines and the vertex is the point where these meet.

During the interview, the students were asked to complete three tasks.

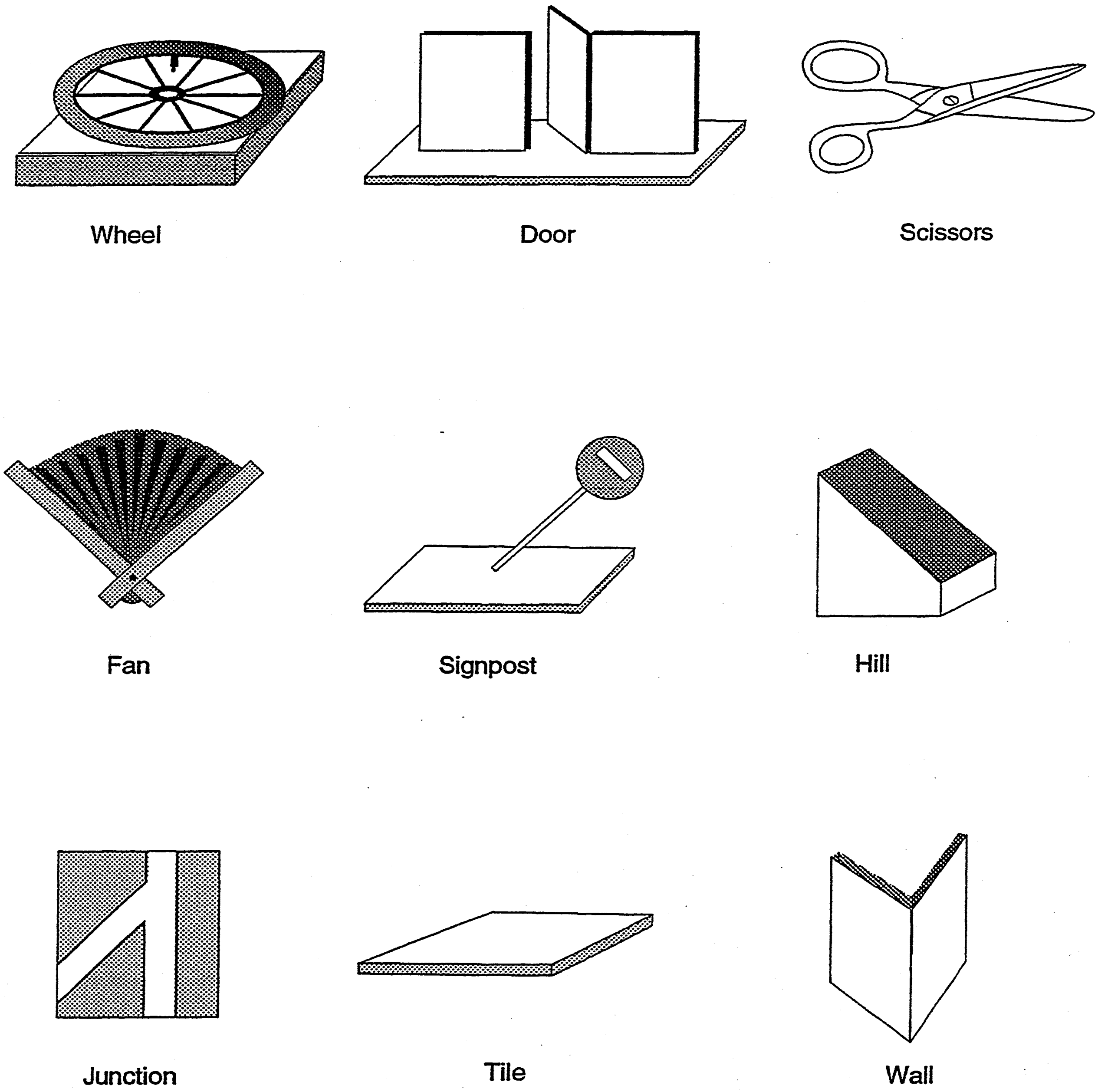


Figure 1: Models used to represent nine physical angle situations

Task 1: Students were asked to show how the abstract angle model could represent the given situation. For example, the student was asked to use the flexible drinking laid on the table in front of her/him to show the turning wheel or a corner of the tile.

Task 2: Students were asked to demonstrate an angle of 45° on the physical model. (The angle of 45° was defined for the student by opening the flexible drinking straw by about 45° and then replacing it by the fixed straw.) For example, students were asked to turn the wheel through 45° or to select the tile (from three possibilities) whose corner had an angle of 45°. For the movable models an error of ±15° was allowed.

Task 3: Students were asked to place the abstract angle model on the physical model to explain how the 45° angle was formed. In scoring the data, some imprecision was ignored. For example, standard modelling was inferred when the vertex was not exactly on the pivot for the scissors or the lines were not exactly parallel to the edge of the roads of the junction.

Each of the nine situations was administered to 48 students in each of grades 2, 4 and 6. (Each student responded to three situations.) When the analysis of these data showed that the wheel, door and hill were still causing difficulty to grade 6 students, these three situations were also administered to 48 students from grade 8.

Results

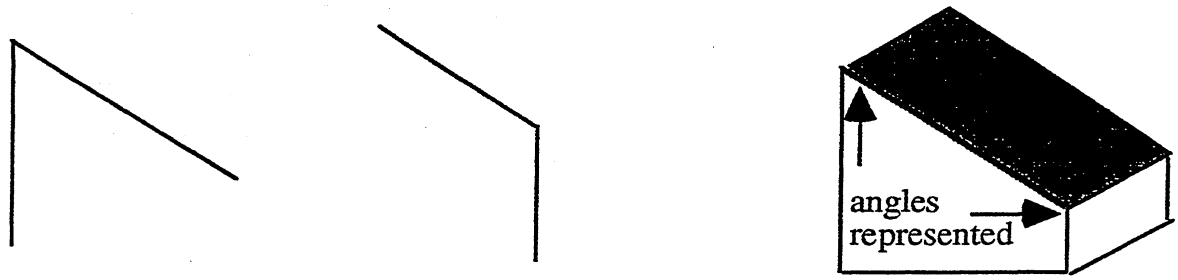
Task 1

The percentages of students who used the abstract model to represent the angular features of each situation in the standard manner are shown in Table 1.

Table 1
Percentage of sample using abstract angle model to represent physical angles in standard manner

Grade	Wheel	Door	Scissors	Fan	Signpost	Hill	Junction	Tile	Wall
2	17	90	100	98	24	48	77	96	96
4	90	96	90	88	67	50	85	100	98
6	94	92	94	92	83	67	96	100	100
8	94	94	—	—	—	59	—	—	—

Mitchelmore and White (1996a) previously noted the difficulties caused by the wheel, signpost and hill for grade 2 students. The difficulties with the wheel largely disappeared by grade 4, and with the signpost by grade 6. However, the hill was still difficult in grade 8. The tendency to represent the hill by a single sloping line (common in grade 2) was rare in grades 4-8. The most common incorrect configuration showed both lines sloping differently in an inverted V-shape. Furthermore, only 19% from grade 4, 27% from grade 6 and 17% from grade 8 represented the angle between the horizontal and the slope of the hill (as most mathematics teachers probably would). The rest of the students who correctly represented the slope using a standard angle preferred to use one of the vertical edges of the hill, as shown in Figure 2. That is, most students opted for a line actually present in the physical model for their second reference line.



Abstract representations for the hill

Hill model

Figure 2

Task 2

The percentages of students who correctly demonstrated an angle of 45° on the physical model are shown in Table 2.

Table 2
Percentage of sample correctly demonstrating size of abstract 45° angle on physical models

Grade	Wheel	Door	Scissors	Fan	Signpost	Hill	Junction	Tile	Wall
2	49	77	87	87	47	43	60	64	64
4	85	100	100	100	88	58	92	96	98
6	98	98	98	98	96	65	96	98	100
8	94	100	—	—	—	75	—	—	—

Most of the difficulties grade 2 students had with this task (as reported in Mitchelmore & White, 1996a) disappeared by grade 4. The exception was the hill, where almost all the errors consisted of choosing the hill with a slope of 22.5°. Students seemed sure that the 45° abstract model represented a sloping hill, but were rather uncertain about how to represent the amount of slope.

Task 3

The percentages of students who could explain how the abstract model matched the physical model are shown in Table 3.

Table 3
Percentages of sample explaining how abstract 45° angle matched physical models

Grade	Wheel	Door	Scissors	Fan	Signpost	Hill	Junction	Tile	Wall
2	21	44	87	79	43	45	94	100	92
4	31	50	89	81	75	60	98	98	92
6	56	65	92	85	88	69	100	96	100
8	42	65	—	—	—	81	—	—	—

By grade 6, over 85% of students could explain how the abstract model matched the angle on the physical model for the scissors, fan, junction, tile and wall. In all of these situations, both arms of the angle are clearly visible. For the signpost, students appear to have learnt between grade 2 and grade 6 to use the horizontal ground as the second arm of the angle. (Very few students modelled the angle between the post and the vertical.) However, even by grade 8, the abstract angle model was not easily matched to the 45° angle in the wheel or door. Performance on the hill improved markedly between grade 2 and grade 8, with most explanations again using visible lines on the physical model (see Figure 2).

Discussion

For each of eight of the nine physical models, more than 80% of grade 6 students used an abstract model to give a standard representation of the implicit angle and more than 95% demonstrated an angle of 45° correctly. The percentage who could explain how the abstract model matched the angle in the physical model was considerably lower in three of the nine situations. It seems safe to make two conclusions:

- by the end of grade 6, the majority of students can *globally* recognise abstract angles (Tasks 1 and 2) in almost all physical situations, but
- even by grade 8 many students cannot *analytically* recognise abstract angles (Task 3) in a number of physical situations—namely, those where the arms of the abstract angle do not match visible lines in the physical model.

The hill appears to be an anomaly. The data for the other eight situations either show a consistent increase across all three tasks (scissors, fan, signpost, junctions, tiles, walls) or a high percentage of students who recognised the abstract angles globally but not analytically (wheel, door). The hill, on the other hand, shows relative low rates of global recognition and relatively high rates of analytic recognition. We argue that this anomaly is caused by a focus on visible lines which accidentally suggest standard modelling. Responses to Tasks 1 and 2 indicate that, although most students have some global concept of slope, many do not quantify slope by relation to a fixed reference line. When such students respond to Task 3 by placing one line of the abstract model on the vertical edge, they could simply be choosing an available visible line to fit the abstract model. Such responses do not necessarily imply any awareness of the significance of the vertical edge as an appropriate reference line. We conjecture that, had the model of the hill been supported by edges which were either not vertical or not visible, far fewer students would have analytically recognised the implicit angle.

Conclusion

The data strongly support the claim that students up to grade 8 use visible lines as the basis for recognising abstract angles in familiar physical situations. Table 4 shows the data in Table 3 rearranged according to the number of visible lines, namely no lines visible (wheel), one line visible (door, hill), or two lines visible (scissors, fan, junction, tile, wall, signpost).

Where the two lines are obvious, children at all age levels have little difficulty using an abstract angle to model the situation. When only one line is visible, students have difficulty imagining the appropriate reference lines to make up an angle. Many are still having problems in grade 8. When no line is visible, students' difficulties are even greater.

We note that two of the three classes in Table 4 contain both movable and fixed situations. Whether a situation is movable or fixed does not seem to be a crucial determinant of the difficulty students have in identifying the abstract angle.

Table 4
Percentage of sample explaining how abstract 45° angle matches physical model, by number of visible lines

Grade	0 lines visible	1 line visible	2 lines visible
2	21	44	82
4	31	55	89
6	56	67	94
8	42	73	—

Implications for teaching

Our results question the argument that teaching the angle concept should be based on the particular physical relationship of turning between the two arms of the angle (Wilson & Adams, 1992). The data suggest turning is not spontaneously conceived as an angle and so it seems unlikely that a focus on turning is a good strategy. A more obvious strategy is to help children recognise the two lines needed to make up the angle and the significance of the relation between them (turning, sharpness, opening or whatever).

The results suggest that it should be possible to start teaching about angle as early as grade 2, provided the examples are limited to situations where both arms of the abstract model are clearly present. Other situations involving angles need to be placed in a hierarchy according to the number of visible lines. The following points are some suggested learning activities. A creative teacher will think of many more.

Situations with two visible lines

- Identify the two lines on scissors, fans, junctions, tiles etc.
- Identify the two lines on leaning objects such as signposts etc.
- Identify angle-related similarities between these situations. For example, place scissors around the corner of a tile or along the two roads which meet at a junction to show that each of these situations involves the same idea.
- Discuss the precise position of the vertex and the lines in some situations. For example, the pivot point on a pair of scissors is not where the cutting edges meet.
- Discuss features peculiar to each physical situation. For example, the sharpness of a tile corner is related to how pointy the corner is, not how thin or fat the tile is.

Situations with one visible line

- Identify the second (imaginary) line in doors and hills.
- Discuss what "horizontal" and "vertical" lines are.
- Link blackboard compasses with a door by showing how the one arm of the compasses is like the closed position of the door and the other arm like the opening door. Thus the imaginary line of a door matches a visible arm in the compass.
- A tile can be compared with a hill. A blackboard ruler can be used as a hill and tiles made or drawn so that some have corners which are not 90°. The angle between the hill and the horizontal is like a corner of a tile. The similarity is that both situations consist of two lines joined at a point.

Situations with no visible lines

- Start with a rotating fan to model the idea of a rotating radius and identify the beginning and finishing position of the radius. Identify also the centre of rotation.

- Model other rotations (wheels, knobs, body turns). Identify how many full turns are executed (two, one, half) and interpret these as a movement from the beginning to the finishing position.
- A flexible abstract model (e.g., using geostrips) can be used to model turns of a wheel by holding one arm fixed and rotating the other.

In conclusion, turning is easy to see in movable angle situations, but the two lines are not obvious. The two lines are more obvious in some fixed situations, but the turning is not easy to see. Students will only have a firm basis for further work on angle if they can identify both the lines and the amount of turn in a variety of familiar angle situations. The aim of early teaching should be to establish this foundation. Without it, it is no wonder that leaning to use a protractor is such a formidable task for many grade 7 and 8 students.

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